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Topic: Electromagnetic field energy density and momentum density

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Electromagnetic energy density

The work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is

$$W_e = \frac{\epsilon_0}{2}\int E^2 \, d\tau,$$

where $\vec{E}$ is the resulting electric field.

The work required to get currents going (against the back e.m.f.) is

$$W_m = \frac{1}{2\mu_0}\int B^2 \, d\tau,$$

where $\vec{B}$ is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields is.

$$U_{em} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau \quad \text{—(1)}$$

Where, the energy per unit volume stored in e.m. field is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \quad \text{--- (2)}$$

In case of monochromatic plane wave

$$B^2 = \frac{1}{c^2} \cdot E^2 = \mu_0 \epsilon_0 \cdot E^2 \quad \text{--- (3)}$$

This shows that the electric and magnetic contributions are equal for electromagnetic waves.

Therefore,

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \epsilon_0 E^2\right)$$

$$u = \epsilon_0 E^2$$

$$u = \epsilon_0 \cdot E_0^2 \cdot \cos^2(kz - \omega t + \delta) \quad \text{--- (4)}$$

As the wave travels, it carries this energy along with it. Now, we will calculate energy flux density.

The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector,

$$S = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \quad \text{—— (5)}$$

for monochromatic plane waves propagating in the z direction

$$\vec{S} = c\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z}$$

$$\boxed{\vec{S} = cu\hat{z}} \quad \text{—— (6)}$$

This eqn represents that energy flux density is equal to the energy density(u) times the velocity of wave ($c\hat{z}$).

Hence, the energy per unit time, per unit area, transported by the wave is

$$\boxed{S = uc}$$

Momentum density

Electromagnetic fields not only carry energy, they also carry momentum.

The momentum density stored in the fields is,

$$\vec{P} = \frac{1}{c^2}\vec{S} \quad \text{—— (7)}$$

For monochromatic plane waves, then

$$\vec{P} = \frac{1}{c}\epsilon_0 \cdot E_0^2 \cos^2(kz - \omega t + \delta)\hat{z}$$

$$\vec{P} = \frac{1}{c}u\hat{z}$$

In case of light, the wavelength is so short ($\sim 5\times10^{-7}m$) and the period is so brief ($\sim 10^{-15}s$), Therefore, we want average value, so, the average of cosine squared over a complete cycle is $\frac{1}{2}$;

$$\langle u \rangle = \frac{1}{2}\epsilon_0 \cdot E_0^2 \quad \text{——— (8)}$$

$$\langle \vec{S} \rangle = \frac{1}{2}c\,\epsilon_0 E_0^2\,\hat{z} \quad \text{——— (9)}$$

$$\langle \vec{P} \rangle = \frac{1}{2c}\epsilon_0 \cdot E_0^2 \cdot \hat{z} \quad \text{——— (10)}$$

The average power per unit area transported by an e.m. wave is called the intensity:

$$I \equiv \langle S \rangle = \frac{1}{2}c\,\epsilon_0 \cdot E_0^2 \quad \text{——— (11)}$$

When light falls on a perfect absorber it delivers its momentum to the surface. In a time Δt the momentum transfer is (fig. 1)

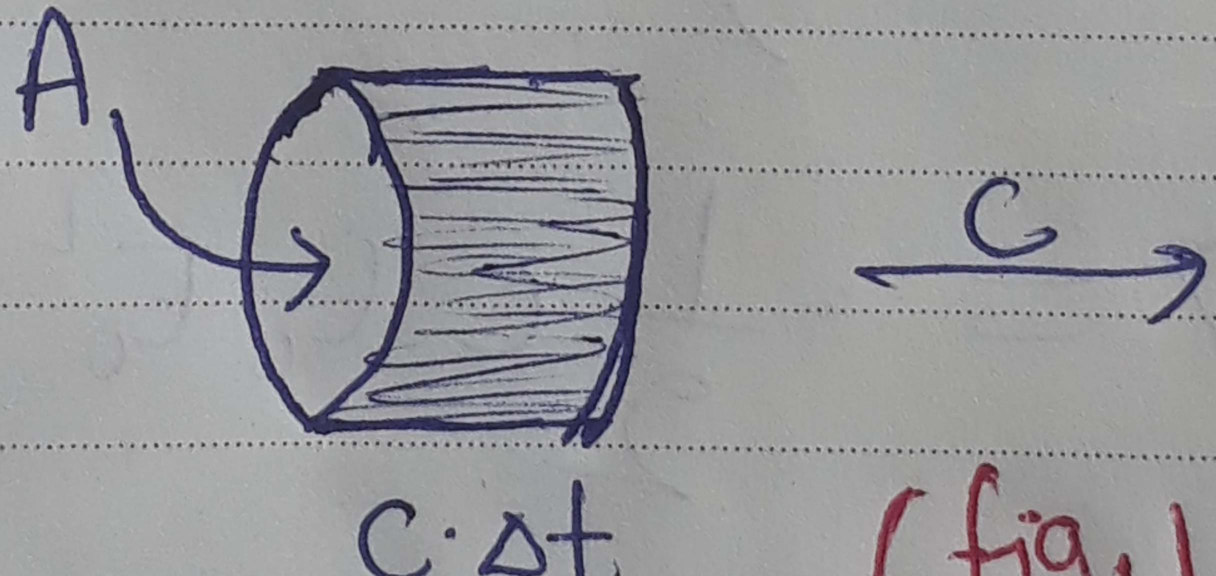
$$\Delta \vec{p} = \langle \vec{P} \rangle A \cdot c \cdot \Delta t, \quad (12)$$

So the radiation pressure (average force per unit area) is:

$$P = \frac{1}{A} \cdot \frac{\Delta p}{\Delta t} = \frac{1}{2} \epsilon_0 \cdot E_0^2$$

(using eq^n (10) & (12))

$$P = \frac{I}{c} \quad (13)$$



(fig. 1)

The net force on all the charges in the surface produces the pressure.